1. DEFINITIONS: The set F from which the n^2 entries of an nxn matrix M are drawn we will call the <u>set of elements of M</u>. For conciseness, by <u>a permutation p of the elements of M</u> we will mean, according to context, either a permutation on F or the transformation of M to a matrix Mp that it induces. For example: taking $F=\{0,1,2,3\}$ and M as shown, p = (01) induces the following transformation:

$$\begin{array}{c} M \\ \begin{array}{|c|c|c|c|} \hline 0&1&2&3 \\ \hline 3&2&1&0 \\ \hline 2&3&0&1 \\ \hline 1&0&3&2 \\ \hline \end{array} \end{array} \longrightarrow \begin{array}{c} Mp \\ \begin{array}{|c|c|c|c|} \hline 1&0&2&3 \\ \hline 3&2&0&1 \\ \hline 2&3&1&0 \\ \hline 0&1&3&2 \\ \hline \end{array} \end{array}$$

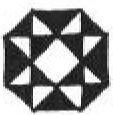
Note that the entries of M occur within a lattice of squares. We will continue to view matrices in this way, whether or not the lattice is shown explicitly.

We now substitute for the elements of M the four triangular half-squares (which we will call simply <u>triangles</u>) in this order:

0 1 2 3

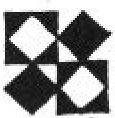
The diamond-like matrix D at left is the result of this substitution.

2. FACT: D has the following remarkable properties:



D

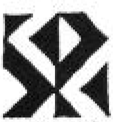
(1) Any matrix (i.e., geometric figure) obtained from D by a permutation of elements is either symmetric or <u>self-complementary</u> (black and white interchanging) under some rigid motion of the square. (Examples at left.)



D(02)

(2) Any matrix obtained from D by a <u>row-column</u> permutation (one of rows and of columns) is also symmetric or self-complementary.

(3) The same is true for permutations of <u>quadrants</u> (the subsquares obtained by bisecting a square along each median).



D(021)

(4) The same is true for any combination of the three sorts of permutations mentioned above.

The author, an artist, discovered these properties in 1975 while devising figures to use in an abstract painting. This monograph is an attempt to explain them and to view D in its proper setting.

3. DEFINITIONS: A <u>geometric matrix</u> is one whose elements are geometric figures -- specifically, subsets (such as triangles) of the square. (We will neglect, when convenient, sets of zero area occurring as subset boundaries.) A <u>foursquare</u> is a 4x4 four-element matrix each of whose elements occurs four times as an entry.* A <u>diamond</u> is a foursquare whose elements are triangles.

*In a general "four-element" matrix, all entries might be identical, but would still be regarded as <u>drawn from</u> a set of four.

The geometric properties of a diamond arise from the interaction of the geometric properties of its abstract structure (its structure as a rectangular array of distinctively labeled, but otherwise undescribed, elements) with the geometric properties of its elements. We shall describe the properties of structure and of elements that make D work. (Keep in mind that every entry of a square geometric matrix, in being carried to another position under a rigid motion of the square, also undergoes that motion. Hence a rigid motion of the matrix may induce a permutation of its elements.) In the case of the matrices we will consider, interesting results occur when we endow them with algebraic, as well as geometric, properties, by regarding their elements as those of a finite field.

4. FACT: In permuting the rows and columns of M, the following (pairwise structurally incongruent -- see 5) abstract structures arise:

```
a b c d    a b c d    a b c d    a b c d    a b c d
d c b a    b a d c    c d a b    c d a b    b a d c
c d a b    d c b a    d c b a    b a d c    c d a b
b a d c    c d a b    b a d c    d c b a    d c b a .
```

Note that the last structure, with first row and first column in standard order, may be regarded as that of the (unlabeled) Cayley table of the four group. In fact, since they result from row-column permutations, all five of the structures are those of tables of the four group. Thus the properties of D may be regarded as properties of this group, one we will encounter often, in various guises.

5. DEFINITIONS: Suppose the elements of a matrix P are drawn from $\{a,b,c,d\}$. The set of boundary lines (subsets of the matrix lattice) separating within the matrix members of $\{x,y\}$ from those of $\{z,w\}$, where $\{x,y,z,w\} = \{a,b,c,d\}$, is called the <u>x-y block map</u> of P, or L(x,y). (Note: L(x,y) = L(z,w).) Thus any four-element matrix has a set of three block maps, one or more of which may be empty maps. The set of maps is useful because it describes what we have been calling the abstract structure of the matrix; the set, like the structure, is unchanged by permutations of elements. (On the other hand, the ordered triple (L(a,b), L(a,c), L(b,c)) describes the matrix closely.*) Four-element matrices A and B are <u>structurally congruent</u> if some rigid motion changes A's maps to B's. We will represent the <u>field F of order four</u> as the set $F = \{0,1,2,3\}$ with these operations:

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

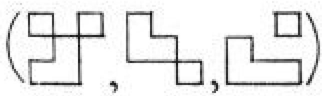
•	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

.

The <u>x-y block matrix</u> B(x,y) of a member P of <u>$M_n(F)$</u> (the set of all nxn matrices over F), where x,y are distinct members of F, is the nxn 0,1 matrix with its 1's where P has x's or y's, its 0's elsewhere.

*If map lines on the edges of the matrix, as well as the interior, are included, then the triple describes the matrix precisely. Example:

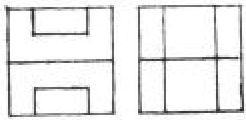
$$\begin{pmatrix} a & d & b \\ c & a & d \\ b & b & c \end{pmatrix} \leftrightarrow$$



6. THEOREM (fundamental): Every member P of $M_D(F)$ is a linear combination of block matrices: $P = 1B(2,3) + 2B(1,3) + 3B(1,2)$.
The proof is easy.* Example:

$$\begin{pmatrix} 0&1&2&3\\ 1&0&3&2\\ 2&3&0&1\\ 3&2&1&0 \end{pmatrix} = 1\begin{pmatrix} 0&0&1&1\\ 0&0&1&1\\ 1&1&0&0\\ 1&1&0&0 \end{pmatrix} + 2\begin{pmatrix} 0&1&0&1\\ 1&0&1&0\\ 0&1&0&1\\ 1&0&1&0 \end{pmatrix} + 3\begin{pmatrix} 0&1&1&0\\ 1&0&0&1\\ 1&0&0&1\\ 0&1&1&0 \end{pmatrix}.$$

7. DEFINITIONS: The <u>complement of a 0,1 matrix</u> is obtained by interchanging its 0's and 1's. (The complement of a subset of the square is its point-set complement.) A square 0,1 matrix (or a subset of the square) is <u>regular</u> if at least one of the following motions of the matrix (the square) leaves the matrix (the subset) unchanged or changes it to its complement: H, a flip about the horizontal median; V, a flip about the vertical median; R^2, 180 degree rotation about the center. A <u>normal</u> block map is one symmetric under H, V, and R^2. A <u>plaid</u> map is a normal one each of whose lines extends all the way across the matrix. (Examples below.)



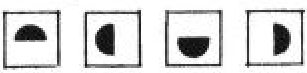
A <u>normal (plaid) four-element matrix</u> is one whose block maps are all normal (all plaid).** (We take the empty map to be plaid.)
A <u>basis map</u> is an nxn, n even, nonempty map that is either plaid and consists of 1 line or of 2 parallel lines, or is a rectangle whose center is that of the matrix.
The <u>sum of two maps</u> A,B is their symmetric difference, i.e. their union minus their intersection.

8. FACT: A sum of normal (of plaid) maps is normal (plaid).

9. FACT: Each block map of a four-element matrix is the sum of the other two block maps. (See 6 and 14.)

10. THEOREM: The set of (plaid) basis maps of nxn, n even, normal four-element matrices is a basis for the vector space of the matrices' normal (plaid) maps. (We take the scalars of the space to be 0 and 1 from F, with products of these and maps the obvious ones.)
PROOF: Clearly the basis maps, normal or plaid, are linearly independent. Let $n = 2m$. By induction on m, one can easily show there are $m^2 + 1$ normal basis maps. They span the space because there are 2 to the $m^2 + 1$ sums (all normal) of basis maps and the same number of normal maps (since each normal map is determined by a quadrant of a block matrix -- or its complement -- and by the presence or absence of each median line). And clearly the plaid basis maps span the space of plaid maps.

11. NOTE: Suppose that in the five structures in part 4, we substitute in any order the following geometric elements:

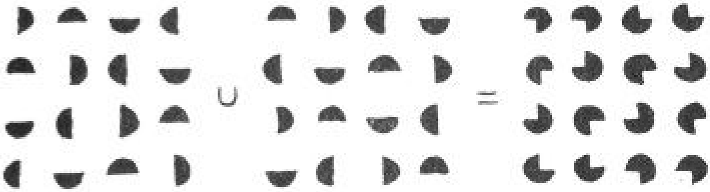
 .

The resulting figures look rather unimpressive until they are superimposed, but then they yield a variety of surprisingly orderly figures.

*First show that $P = 1B(0,1) + 2B(0,2) + 3B(0,3)$. The form above for P was chosen so that each block matrix, like each block map, would be the sum of the other two.

** There are 57 varieties of normal foursquares, modulo structural congruence: 21 plaid, 36 non-plaid.

For example:



The following theorem explains this behavior.

12. THEOREM: Let $A = \{(0,0),(1,1),(2,2),(3,3)\}$,
$B = \{(1,0),(0,1),(3,2),(2,3)\}$,
$C = \{(2,0),(3,1),(0,2),(1,3)\}$,
$D = \{(3,0),(2,1),(1,2),(0,3)\}$.

When two members of $M_n(F)$ are superimposed, we may regard the result, using the above sets, as a matrix with elements A,B,C,D, since when two entries are superimposed, they fall into exactly one of these sets. This new matrix is normal (plaid) if the superimposed matrices are both normal (plaid).

PROOF: Table 1 at left below summarizes how the old entries combine to produce the new. It strongly suggests that we replace letters with numbers; since this has no effect on the truth or falsity of the theorem, we do so, obtaining table 2. Thus the superimposition of matrices is described by their matrix addition. We complete this proof, therefore, by stating and proving a new theorem.

	0	1	2	3
0	A	B	C	D
1	B	A	D	C
2	C	D	A	B
3	D	C	B	A

Table 1

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Table 2

13. THEOREM: A sum of normal (plaid) members of $M_n(F)$ is normal (plaid).

PROOF: Since every normal (plaid) member of $M_n(F)$ is a linear combination of normal (plaid) block matrices, it suffices to prove the theorem for these. The following theorem does so, completing the proof of this theorem and therefore of the one above.

14. THEOREM: The x-y block map of the sum of two block matrices is the sum of their x-y block maps.

PROOF: Each 0,1 matrix has one empty and two identical maps -- the latter showing the lines separating 0's from 1's.

The table at right shows how adjacent entries, and also the map lines (represented by slashes) that may or may not appear between them, are added. Examination of the table shows that the theorem is true.

	0 0	0/1	1/0	1 1
0 0	0 0	0/1	1/0	1 1
0/1	0/1	0 0	1 1	1/0
1/0	1/0	1 1	0 0	0/1
1 1	1 1	1/0	0/1	0 0

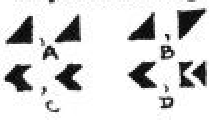
Note that members of $M_n(F)$, regarded as ordered triples of block maps, may be added by adding the triples elementwise.

15. THEOREM: The normal (plaid) members of $M_n(F)$ are a linear algebra of singular matrices over F, if n is even.
PROOF: A <u>basis</u> <u>matrix</u> is a 0,1 matrix whose nonempty maps are basis maps. By 6, 10, and 13, each normal (plaid) member of $M_n(F)$ is a linear combination of normal (plaid) basis matrices. To see that products of such matrices are normal (plaid), note that the product of any two is the zero matrix unless the nonempty maps of the factor on the right are each the horizontal median line, or those of the left factor, the vertical. In such cases, the product is easily shown to be plaid. Therefore the normal (plaid) matrices of $M_n(F)$ are closed under multiplication. We have shown that they are closed under addition, and clearly they are closed under multiplication by scalars. To show singularity of each normal matrix P, express it as a linear combination of basis matrices and multiply it on the left by the 1xn 0,1 vector whose 1's are the central four entries. The result is the zero vector, so P has a nontrivial null space.

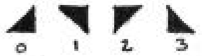
16. DEFINITIONS: A <u>regularity</u> is one of the following six properties: symmetry or self-complementarity under H, V, or R^2. A <u>regular transformation</u> is H, V, R^2, or any of these followed by complementation. A <u>good set</u> of four black and white geometric figures, considered as subsets of the square, is one with the following properties: (1) If one of the figures has a regularity, the other three have the same regularity; (2) If a regular transformation interchanges two figures, it interchanges the other two also; (3) Any 2x2 normal matrix of elements from the set is regular.
By an <u>orbit in a lattice</u> of squares we mean a set of four 1x1 subsquares carried into one another by H, V, and R^2; by <u>an orbit in a matrix</u> we mean a 2x2 submatrix, its entries those in an orbit of the matrix lattice.

17. PROBLEM: Show that the definition of a good set above is redundant since it is equivalent to the following: a "good set" of four distinct figures is one which always yields a regular result when the figures are used in a 2x2 matrix with four distinct entries.

18. THEOREM: Every good set, when substituted for the elements of a four-element matrix (foursquare) P of even order (i.e., 2mx2m), always yields a regular figure if (if and only if) P is normal.
PROOF: Suppose P is normal. Then H, V, and R^2 induce permutations of the elements of P; furthermore, if two elements are transposed by such a permutation, so are the other two. These properties of P, along with properties (1) and (2) of an arbitrary good set, ensure that each orbit of the new matrix shares the regularity of its center guaranteed by property (3). Hence the new matrix is regular.
Now suppose P is a foursquare and not normal. We will first show that it must be structurally symmetric (i.e., its set of block maps must be invariant) under H, V, and R^2 if it is to enjoy the property described in the theorem for <u>all</u> good sets. What happens if we replace its elements with triangles? Suppose it lacks structural symmetry under H or V -- say, without loss of generality, H, since triangles remain triangles under a 90 degree rotation. Then it must also lack symmetry under V or R^2, since $H = VR^2$. If it lacks R^2 symmetry, the only possible regularity is under V; then use one of pairs A or B below as the middle two entries in some row to preclude regularity. If it lacks V symmetry, then use one of the pairs C or D in opposite corners of the matrix to preclude regularity. (Pair D and their images under V constitute the good set of what we will call <u>arrows</u>.)



We now will assume that P is structurally symmetric under H, V, and R^2, although not normal, and also assume that P always yields regular figures under substitution of triangles. This will enable us to eliminate from consideration all but eight sorts of matrices, each of which will not work when arrows are used. Now, since P is not normal, under either H or V -- say H -- some map must be asymmetric. This implies that H, acting as a permutation on P's elements, is of the form (ab)(c)(d) -- say, taking the elements to be 0,1,2,3, that H = (01)(2)(3) = (01). (If H were, say, (01)(23), then it would interchange the sets {0,2}, {1,3} and also the sets {0,3}, {1,2}, so all three block maps would be symmetric under H, since if two are the third must be also. And clearly H is neither a 3-cycle nor a 4-cycle nor the identity permutation.) Now suppose, on the one hand, that there exists some triangle matrix formed from P with a V regularity. Then V is either (01)(23) or (02)(13) or (03)(12). (For a triangle matrix, there is no regularity under V if any element is taken to itself by V acting as a permutation.) Following H, V must give either (01)(01)(23) = (23) for R^2 or give a 4-cycle for R^2, which is impossible. Hence H = (01), V = (01)(23), and R^2 = (23). As 2 and 3 are invariant under H, there can be no regularity of a triangle matrix under H. And the following ordering of elements produces a figure with regularity under neither V nor R^2:



Suppose, on the other hand, that no triangle matrix formed from P has a V regularity. Then V cannot be of the form (ab)(cd), so must be the identity I or of the form (ab).

If V = I, then R^2 = (01), so let 0 be the triangle with 90 degree angle at lower left and 1 be the triangle with 90 degree angle at lower right to get a figure with no regularities. If V is of the form (ab), then V = (01) or (23), since any other choice for V makes R^2 = HV a 3-cycle, which is impossible. If V = (23) then R^2 = (01)(23), so again let 0 and 1 be triangles with 90 degree angles as described to get a figure with no regularities. If V is (01) then R^2 = I, and P must be structurally congruent to one of the following eight matrices:

$$
\begin{matrix}0&0&1&1\\2&3&3&2\\2&3&3&2\\1&1&0&0\end{matrix}\qquad
\begin{matrix}0&1&0&1\\2&3&3&2\\2&3&3&2\\1&0&1&0\end{matrix}\qquad
\begin{matrix}0&2&2&1\\3&0&1&3\\3&1&0&3\\1&2&2&0\end{matrix}\qquad
\begin{matrix}0&2&2&1\\3&1&0&3\\3&0&1&3\\1&2&2&0\end{matrix}
$$

$$
\begin{matrix}2&0&1&2\\0&3&3&1\\1&3&3&0\\2&1&0&2\end{matrix}\qquad
\begin{matrix}2&0&1&2\\3&0&1&3\\3&1&0&3\\2&1&0&2\end{matrix}\qquad
\begin{matrix}2&0&1&2\\3&1&0&3\\3&0&1&3\\2&1&0&2\end{matrix}\qquad
\begin{matrix}2&0&1&2\\1&3&3&0\\0&3&3&1\\2&1&0&2\end{matrix}\ .
$$

An irregular arrow matrix may be formed from each of the eight, since each has one orbit entirely of 3's and one with 0's on the main diagonal and 1's on the secondary; these orbits can be

 and 

This completes the proof of the theorem.

19. DEFINITIONS: We need some names to describe the sort of matrices constructed in the proof of the last theorem, at the end. We will call foursquares structurally congruent to any of these eight matrices *pi foursquares*. In general, we will call matrices for which R^2 sends each element to itself *rotal*, and non-normal rotal matrices *skew*. There are 32 skew foursquares, modulo structural congruence. It is convenient to have names for some other sorts of foursquares: The *cyclic* foursquares are those structurally congruent to any of the following, obtained by subjecting the table of the cyclic group of order four to row-column permutations:

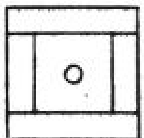
0 1 2 3	0 1 2 3	0 1 2 3	0 1 2 3	0 1 2 3	0 1 2 3	0 1 2 3
1 2 3 0	1 0 3 2	1 3 0 2	1 2 3 0	2 3 0 1	2 0 3 1	2 3 1 0
2 3 0 1	2 3 1 0	2 0 3 1	3 0 1 2	1 2 3 0	1 3 0 2	3 2 0 1
3 0 1 2	3 2 0 1	3 2 1 0	2 3 0 1	3 0 1 2	3 2 1 0	1 0 3 2 .

The *quadratic* foursquares are those (previously given) similarly obtained from the quadratic group (i.e., the four group).

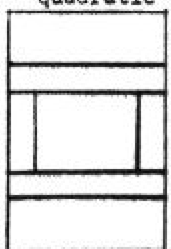
20. FACT: Every 4x4 Latin square is cyclic or quadratic. (To show this, it is sufficient to consider only the squares with first row and column in standard order.)

21. NOTE: Cyclic and quadratic foursquares are closely related in structure. The sketch for a painting in illustration 2 shows this by converting them to triangle matrices in every possible way, modulo matrix rotations. The larger matrices are quadratic, the smaller cyclic. The right half of the sketch is obtained by flipping the left half about its right edge, then rotating each triangle through a 90-degree turn to obtain the *duals* of the flipped matrices. Two of the matrices are *self-dual*, i.e., congruent to their duals; it happens that these are the only self-dual normal triangle foursquares, so dualizing makes the listing of such matrices a good deal easier. Note that in dualizing, axes of symmetry become axes of complementarity, and vice-versa.

The diagrams at right show the regions into which structurally congruent matrices fall, for a given half (left or right) of the sketch.



quadratic



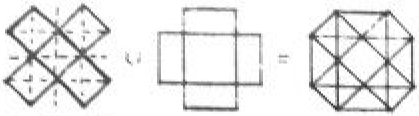
cyclic

22. THEOREM: A triangle matrix is plaid if and only if its diagonal boundary lines between black and white, regarded as elements of a matrix with two sorts of entries, form a plaid matrix, and its horizontal and vertical boundary lines between black and white (its *cuts*) are those of a plaid map.

PROOF: Suppose the matrix is plaid. One sort of diagonal belongs to elements 0 and 2 at left, the

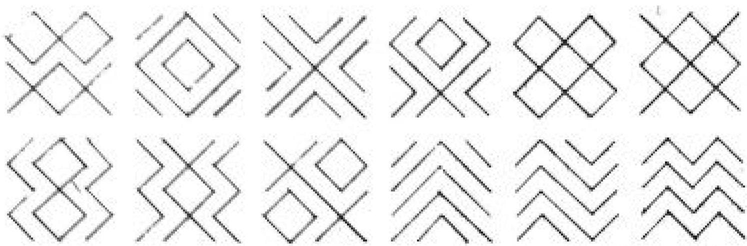
other to 1 and 3. Since the 0-2 map is plaid, the condition on diagonals is satisfied. A vertical cut can appear only between 0 and 1 or between 2 and 3. Thus the vertical cuts are the complement, in the set of possible lines, of the vertical lines of the 0-1 map, hence are those of a plaid map; similarly for horizontal cuts. If the boundary-line conditions are satisfied, then the same line of reasoning shows that the 0-2 map is plaid, and that the vertical (horizontal) lines of the 0-1 (0-3) map are those of a plaid map. Hence, since each map is the sum of the other two, all three maps are plaid.

23. NOTE: The theorem above provides an easy way of listing all nxn plaid triangle matrices: "cut" with plaid maps the diagonal-line figures obtained by "coloring" plaid maps with diagonal lines, then deleting the maps. For example:



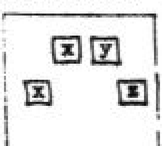
← Cutting the diamond D.

The 12 diagonal-line figures used in cutting diamonds are:



Note that one occasionally obtains a non-foursquare by cutting one of these figures with a plaid map; such non-foursquares, and those obtained by cutting the figure at left, have regularity properties like those of foursquares. There are, modulo rigid motions and complementation, 143 plaid diamonds, and 29 plaid triangle matrices that are not foursquares.

24. THEOREM: A 2nx2n (4x4) triangle matrix is normal (plaid) if and only if each orbit (quadrant) has the same set of block maps and the same nonempty set of regularities.
PROOF: If a 2nx2n matrix is normal, then (by the proof of 18) each orbit has the same nonempty set of regularities; if two of the orbits differ in structure, there must be an exchange of elements under H or V of the sort impossible in a normal matrix. Now assume that the orbits satisfy the given conditions. If the square is not normal, then either its set of block maps changes under H or V, or it does not and H or V induces a permutation of elements of the form (ab)(c)(d). In the former case, we have a situation like that shown at left, which is ruled out by a consideration of all combinations of triangles that may occur in such a situation, since each combination violates the condition on regularities of orbits.



In the latter case, suppose V, say, induces, say, (ab)(c)(d).

$$\begin{matrix} a & b \\ c & d \end{matrix} \quad \begin{matrix} a & b \\ b & a \end{matrix} \quad \begin{matrix} a & b \\ a & b \end{matrix}$$

The orbits in which a and b are switched must look like one of the three at left. In each case, it is impossible for c and d to remain unswitched since the orbits in which they occur must look the same way. The similar result for 4x4 plaid matrices and their quadrants follows from the fact, easily verified, that a 2x2 triangle matrix is regular if and only if it is plaid, from the fact that by row-column permutations a quadrant can be made the central orbit of a new 4x4 matrix whose orbits are the quadrants of the old, and from the following theorem.

25. THEOREM: The set of 4x4 plaid matrices (foursquares) is closed (closed and transitive) under permutations of rows, columns, and quadrants.

PROOF: To show closure, it is sufficient to show that 4x4 plaid basis matrices remain plaid under such permutations, and this is easily done. To show transitivity for foursquares, one may use brute force to find, modulo permutations of elements, the (6) subsets transitive under row-column permutations and to show that a member of each is taken by row-column-quadrant permutations to the matrix at left, for which quadrant permutations induce all possible permutations of elements.

$$\begin{matrix} a & a & b & b \\ a & a & b & b \\ c & c & d & d \\ c & c & d & d \end{matrix}$$

26. PROBLEM: Show that the set of block maps of a foursquare remains invariant under row or column permutations of the four-group type (i.e., I, (12)(34), (13)(24), (14)(23)) if and only if the matrix is plaid.

27. PROBLEM: Show that a normal foursquare remains normal under all permutations of rows and columns only if it is plaid.

28. MISCELLANEOUS PROBLEMS:

(1) The four group is the only one that occurs as a normal subgroup of an alternating group. List other unique properties of the four group.

(2) Prove, or disprove, modify, and prove the modification: If one block matrix of a diamond is obtained from a normal block matrix by row-column-quadrant permutations, and another is plaid, then (a)-each quadrant is normal or each is non-normal; (b)-some regular transformation T combined with a row-column permutation P, each acting on quadrants rather than on the whole matrix, converts one of each of two pairs of quadrants to the other of the pair. What sets of such diamonds are transitive under row-column-quadrant permutations? Generally, what diamonds have properties a and b?

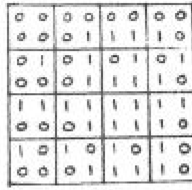
(3) Which-if any- of the results we have considered have analogues for 8x8 matrices? Do any of the groups of order 8 have properties in some way analogous to those of the four group?

(4) Which-if any- of our results for even-order matrices have analogues for odd-order matrices?

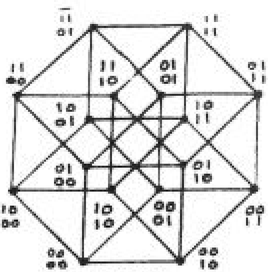
(5) What - if any- are the interesting algebraic properties of the various classes, other than normal, of matrices that we have mentioned: cyclic, rotal, skew, pi, and those in problem 2 above?

(6) Prove or supply a counterexample: If $B_i(N)$ is the 0-i block matrix of N, then for normal foursquares P, Q we have $PQ = 1^2B_1(P)B_1(Q)+2^2B_2(P)B_2(Q)+3^2B_3(P)B_3(Q)$, considering P,Q in $M_4(F)$.

29. NOTE: In analyzing the structure of a 4x4 matrix, the tesseract (i.e., hypercube) T at right below is sometimes useful. Note that T is the diagram showing partial ordering, by inclusion, of the subsets of a four-element set. If the matrix Q at the right is considered as drawn on a torus, then entries of Q are adjacent if and only if the corresponding vertices of T are. The entries in a row, column, or 2x2 submatrix of Q correspond to the vertices in a parallelogram in the drawing of T.

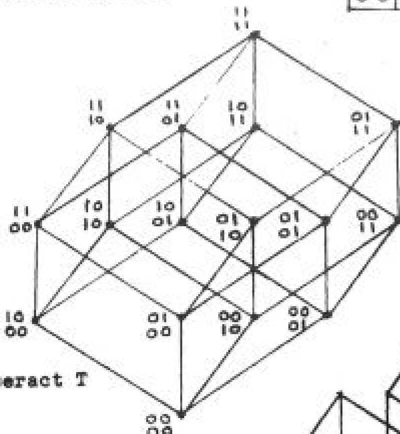


Q

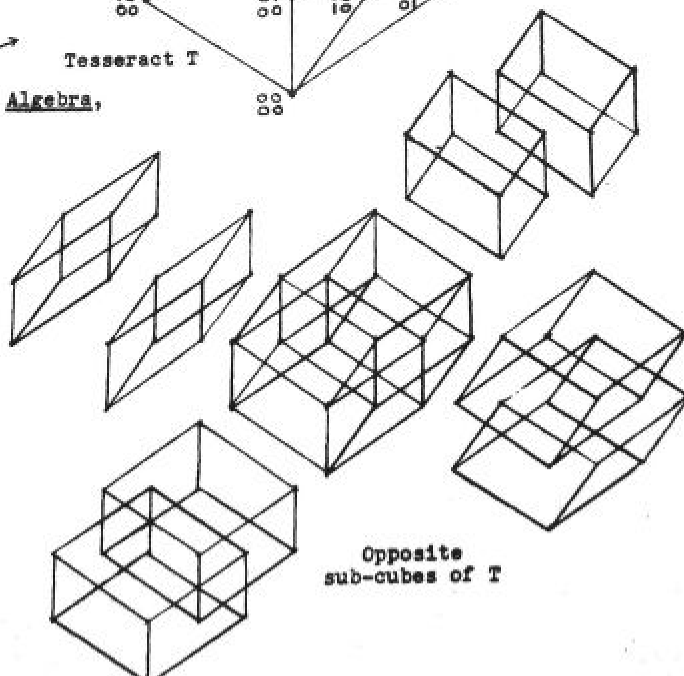


Symmetric Version of T

Reference: F. E. Hohn, <u>Applied Boolean Algebra</u>, Macmillan, 1960



Tesseract T



Opposite sub-cubes of T

30. NOTE: Listed below are the various ways in which rigid motions of the square induce permutations of entries, for the sorts of foursquares we have mentioned. By R we mean a 90-degree counter-clockwise rotation; by D, a flip about the principal diagonal; by D', a flip about the secondary diagonal.

	R^2	V	H	R	R^{-1}	D	D'
normal:							
1	I	I	I	(ab)	(ab)	(ab)	(ab)
2	I	(ab)(cd)	(ab)(cd)	(ab)(cd)	(ab)(cd)	I	I
3	I	(ab)(cd)	(ab)(cd)	(ab)	(ab)	(cd)	(cd)
4	I	(ab)(cd)	(ab)(cd)	----	----	----	----
5	(ab)(cd)	I	(ab)(cd)	----	----	----	----
6	(ab)(cd)	(ad)(bc)	(ac)(bd)	(acbd)	(adbc)	(cd)	(ab)
7	(ab)(cd)	(ad)(bc)	(ac)(bd)	----	----	----	----
pi:							
8	I	(ab)	(ab)	I	I	(ab)(cd)	(ab)(cd)
9	I	(ab)	(ab)	(ab)(cd)	(ab)(cd)	(cd)	(cd)
10	I	(ab)	(ab)	(ab)	(ab)	I	I
11	I	(ab)	(ab)	----	----	----	----
skew, non-pi:							
12	I	----	----	(ab)(cd)	(ab)(cd)	----	----
13	I	----	----	(ab)	(ab)	----	----
14	I	----	----	----	----	----	----
cyclic, non-normal:							
15	(ab)	----	----	----	----	I	(ab)
16	(ab)	----	----	----	----	----	----
17	----	(ab)(cd)	----	----	----	----	----

If we use the Polya-Burnside theorem to enumerate the geometric matrices that can be formed from two foursquares that fall into the same one of the 17 categories above, the result for each foursquare will be the same. Furthermore, the geometric matrices formed from one foursquare will resemble those formed from the other, as far as symmetry and self-complementarity are concerned.

For reference, here is the group of symmetries of the square:

	I	R^2	V	H	R	R^{-1}	D	D'
I	I	R^2	V	H	R	R^{-1}	D	D'
R^2	R^2	I	H	V	R^{-1}	R	D'	D
V	V	H	I	R^2	D	D'	R	R^{-1}
H	H	V	R^2	I	D'	D	R^{-1}	R
R	R	R^{-1}	D'	D	R^2	I	V	H
R^{-1}	R^{-1}	R	D	D'	I	R^2	H	V
D	D	D'	R^{-1}	R	H	V	I	R^2
D'	D'	D	R	R^{-1}	V	H	R^2	I

ADDENDUM

Generalized Matrix Multiplication

In computing the product MN of two 4x4 matrices, we may take the orthogonal foursquares in fig. A below as a guide. To compute an entry of the product, we look at the entry in the same position in A_1; suppose it is n. There are four n's in A_1; we take the entries in M that are in the same positions as the n's in A_1 as the entries of a 1x4 vector, using the order of the entries in the corresponding positions in A_2 as a guide for the order in which we take the entries from M in forming the vector. We then form another vector, from entries in N, now using A_2 as a guide to position and A_1 as a guide to order. Finally, we take the inner product of the two vectors as our entry of MN.

In general, we can use any two orthogonal foursquares to define a multiplication, by using the above procedure -- for instance, we can use those in figs. B or C below. (See examples below.)

Fig. A

$$\underset{A_1}{\begin{matrix} 1&1&1&1\\ 2&2&2&2\\ 3&3&3&3\\ 4&4&4&4 \end{matrix}} \qquad \underset{A_2}{\begin{matrix} 1&2&3&4\\ 1&2&3&4\\ 1&2&3&4\\ 1&2&3&4 \end{matrix}}$$

Fig. B

$$\underset{B_1}{\begin{matrix} 1&1&2&2\\ 1&1&2&2\\ 3&3&4&4\\ 3&3&4&4 \end{matrix}} \qquad \underset{B_2}{\begin{matrix} 1&3&3&1\\ 4&2&2&4\\ 4&2&2&4\\ 1&3&3&1 \end{matrix}}$$

Fig. C

$$\underset{C_1}{\begin{matrix} 1&2&3&4\\ 2&1&4&3\\ 3&4&1&2\\ 4&3&2&1 \end{matrix}} \qquad \underset{C_2}{\begin{matrix} 1&2&3&4\\ 4&3&2&1\\ 2&1&4&3\\ 3&4&1&2 \end{matrix}}$$

Examples: (Entries in matrices are from F.)

Using Fig. B:

$$\begin{pmatrix} 0&1&2&3\\ 1&0&3&2\\ 2&3&0&1\\ 3&2&1&0 \end{pmatrix} \otimes_b \begin{pmatrix} 1&2&3&0\\ 2&0&1&3\\ 3&1&0&2\\ 0&3&2&1 \end{pmatrix} = \begin{pmatrix} 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1 \end{pmatrix}$$

Using Fig. C:

$$\begin{pmatrix} 0&1&2&3\\ 1&0&3&2\\ 2&3&0&1\\ 3&2&1&0 \end{pmatrix} \otimes_c \begin{pmatrix} 1&2&3&0\\ 2&0&1&3\\ 3&1&0&2\\ 0&3&2&1 \end{pmatrix} = \begin{pmatrix} 0&1&2&3\\ 1&0&3&2\\ 2&3&0&1\\ 3&2&1&0 \end{pmatrix}$$

Using Fig. C:

$$\begin{pmatrix} 0&0&1&1\\ 0&0&1&1\\ 2&2&3&3\\ 2&2&3&3 \end{pmatrix} \otimes_c \begin{pmatrix} 1&2&3&0\\ 2&0&1&3\\ 3&1&0&2\\ 0&3&2&1 \end{pmatrix} = \begin{pmatrix} 2&1&0&3\\ 2&1&0&3\\ 3&0&1&2\\ 3&0&1&2 \end{pmatrix}$$

Problem: What sorts of orthogonal pairs of foursquares, besides those in fig. A above, yield multiplications under which the set of normal (or of plaid) members of $M_4(F)$ is closed?

Problem: When is this sort of generalized multiplication associative?