Desargues's Theorem in Light of Galois Geometry

Steven H. Cullinane

August 2013

H. F. Baker's frontispiece to Volume I of his *Principles of Geometry* (Cambridge U. Press, 1922) shows his version of Desargues's theorem. Baker's frontispiece illustration has three triangles, not two, in perspective both centrally and axially.

The fact that the illustration has 15 points and 20 lines suggests a look at the same configuration in a Galois geometry: the finite projective 3-space $PG(3, 2)$.

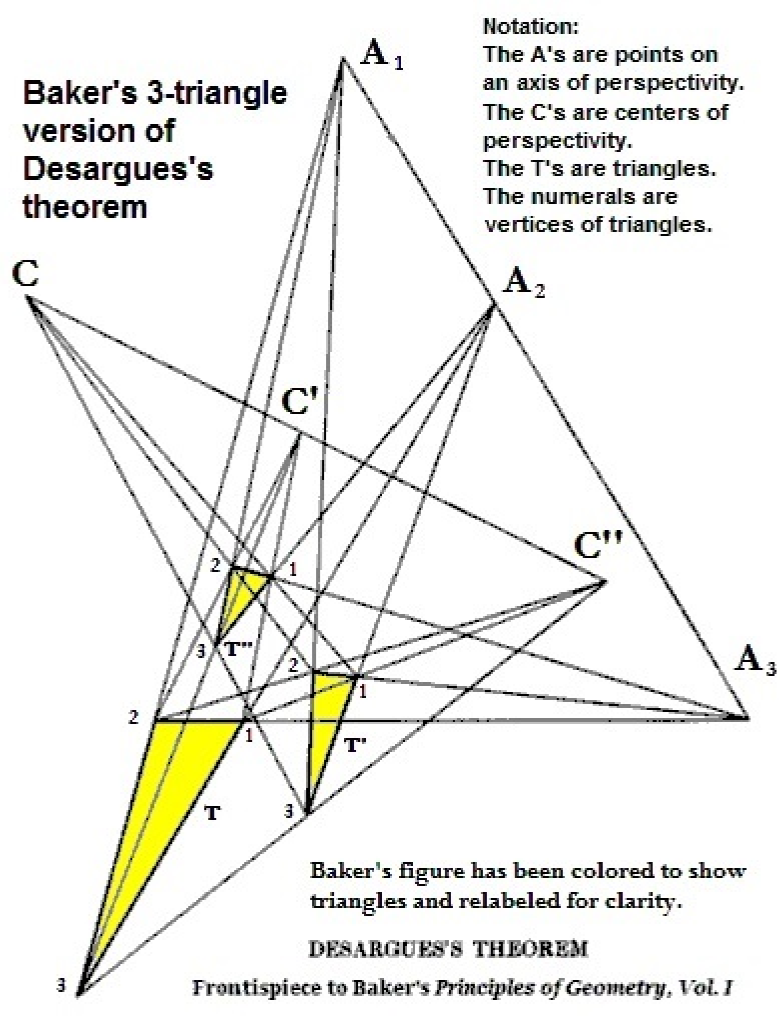

Figure 1: Frontispiece to Baker's 1922 *Principles of Geometry, Vol. I.*

The 15 points of a finite geometry, the Galois 3-space $PG(3,2)$:

	A_1	A_2	A_3
C	V_1	V_2	V_3
C'	V_1'	V_2'	V_3'
C''	V_1''	V_2''	V_3''

Figure 2: All 15 points of $PG(3,2)$.

Here the V's denote vertices of three triangles, the A's denote points on an axis of perspectivity, and the C's denote centers of perspectivity (which, in the 3-triangle version of Desargues's theorem, lie together on a line).

Point-line incidence in the $PG(3,2)$ version of Desargues's theorem is given by the diagrams below, which show the 20 lines from Baker's figure above as 20 of the *Rosenhain tetrads* from Hudson's 1905 classic *Kummer's Quartic Surface*.

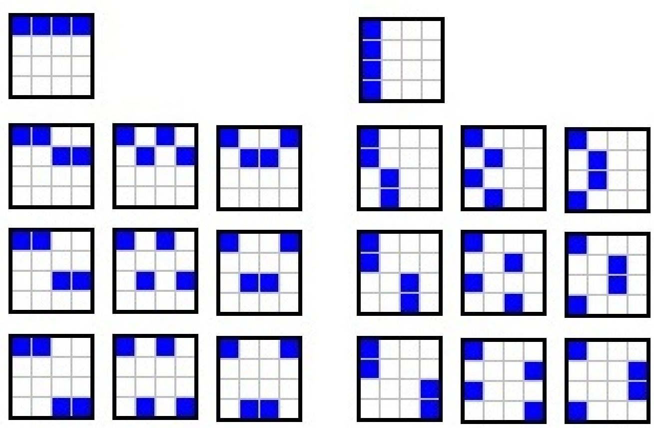

Figure 3: 20 of the 35 lines of $PG(3,2)$.

These 20 Rosenhain tetrads are in fact 20 of the 35 planes through the origin in a finite vector 4-space. The geometry of this 4-space underlies the *Cullinane diamond theorem*. (See, for instance, the article in Springer's online Encyclopedia of Mathematics).

Planes through the origin in a vector 4-space may also be regarded as lines in a projective 3-space—in this case, the Galois projective 3-space $PG(3,2)$ over $GF(2)$. (The origin, at upper left of each box, is disregarded when viewing the twenty 4-point *vector-space* planes through the origin as twenty 3-point *projective* lines.)

Another classical point-line configuration, the Cremona-Richmond figure of 15 points and 15 lines, may be similarly pictured using 15 of Hudson's *Göpel* tetrads.

Here is the frontispiece to Baker's *Principles of Geometry, Vol. IV*:

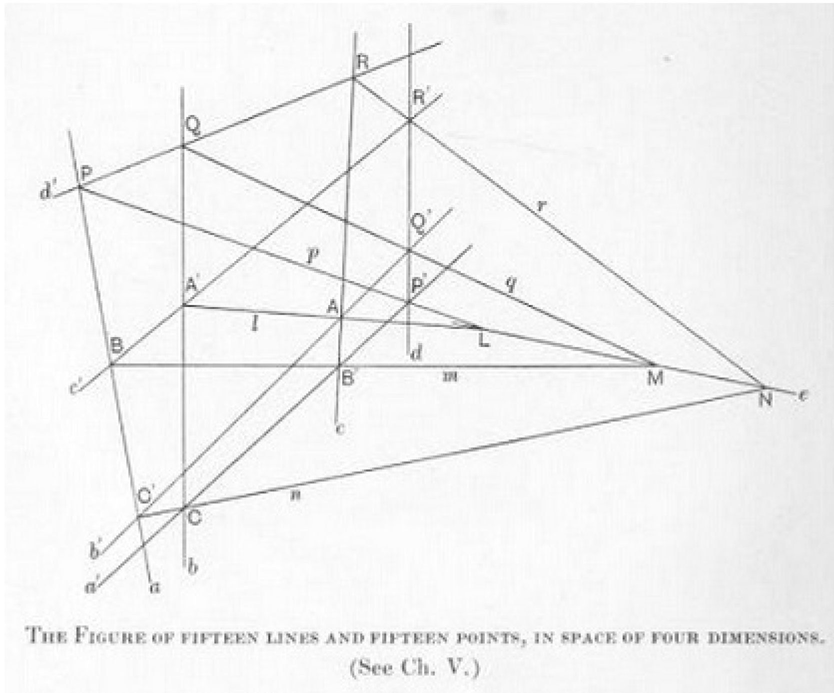

Figure 4: A version of the Cremona-Richmond configuration

A Galois-geometry key to that configuration:

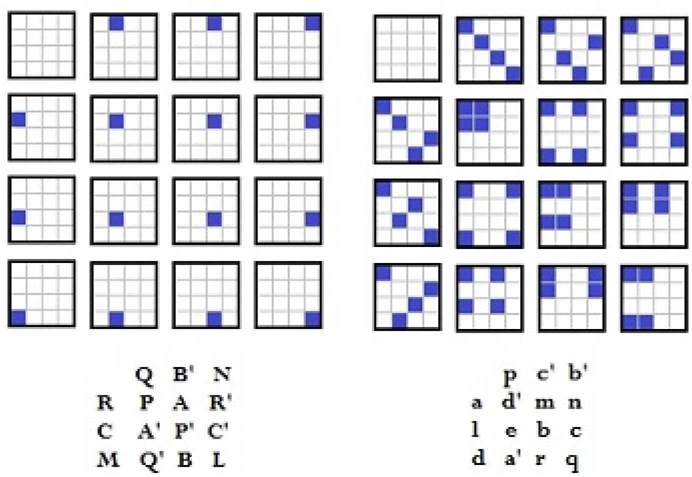

Figure 5: Fifteen points and fifteen lines of $PG(3, 2)$