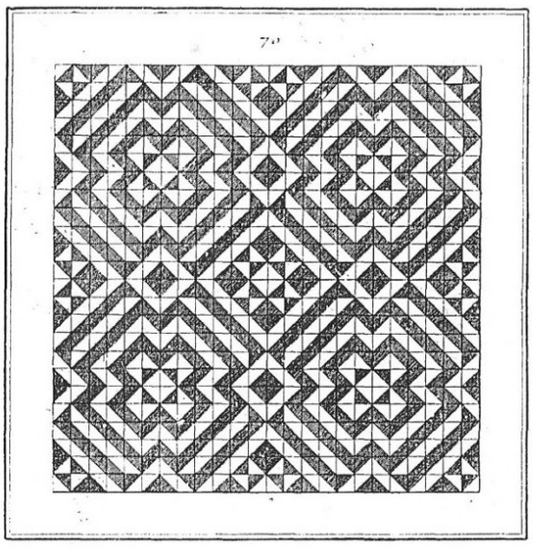
MATHEMOTION

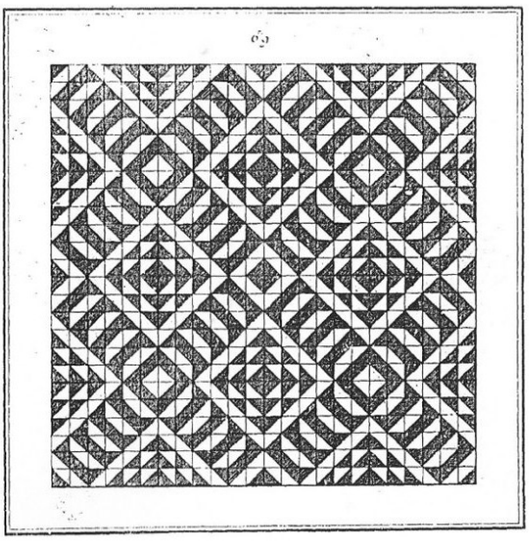
The Truchet Tiles and the Diamond Puzzle

The study of tiling or tessellation patterns is an interesting topic with interest both in the field of science and technology , as well as art and design . In today's entry of the *Scientific Culture Notebook* we are going to focus on **tessellation patterns with a single type of tile** –or tessera-, but which is also extremely simple, the one known as "Truchet tile". It is the square tile divided diagonally into two triangular areas of two different colors, for example, gray and black, like the one that appears in the image.

Truchet tile

In the book *The Sense of Order* (1979) by the Austrian-born British art historian Ernst H. Gombrich (1909-2001), known among other works for his famous *History of Art* (1950), some of the pattern designs are reproduced of tessellations made with this tile that appear in the book *Method for making an infinity of different designs with squares of two colors separated by a diagonal line* (1722), by the French Carmelite father Dominique Doüat (18th century).





Illustrations of tiling patterns, with a Truchet tile, from the book *Method for Making an Infinity of Distinct Designs with Squares of Two Colors Separated by a Diagonal Line* (1722), by Dominique Doüat

A comment

Posted on April 6, 2022 in mathemotion

math

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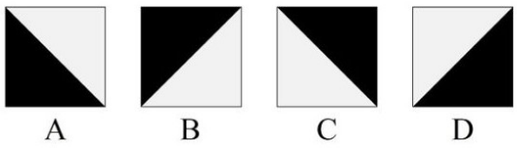
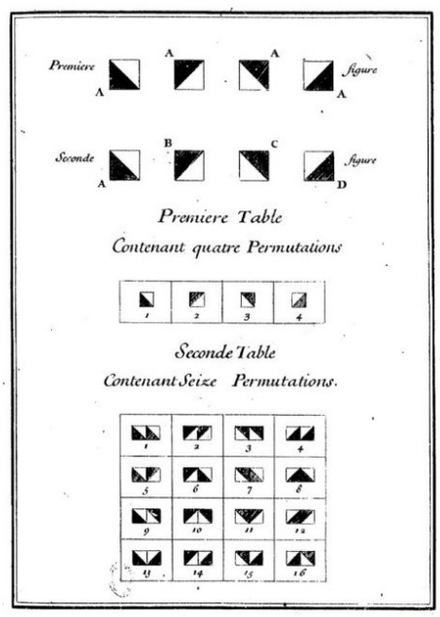
These are just two of the **seventy-two tiling pattern designs included in Doüat's book** (see the bibliography for a link to a facsimile edition with extracts and another to the complete work that can be consulted), which is a text in the that a combinatorial analysis of the tiling that can be generated with that tile is carried out. Father Doüat begins by naming the four orientations of the tile as A, B, C and D, as shown in the image.



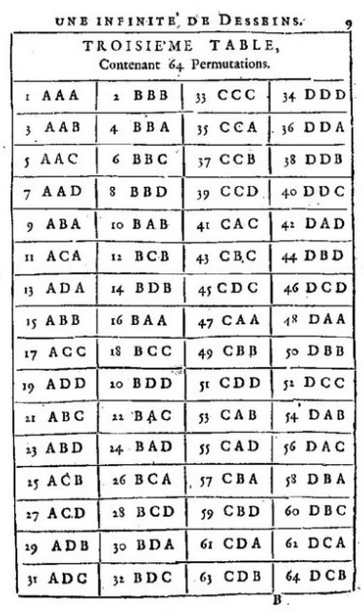
The four possible orientations of the Truchet tile, named by Doüat as A, B, C and D

From that moment on, the Carmelite dedicated himself to carrying out **a combinatorial analysis**with the letters A, B, C and D, starting with the simplest cases, which is the only way to classify in a certain order, between pages 20 and 189 (on many occasions they are pages with lists of letters). For example, the possible tiling with only two tiles (horizontally) would be sixteen (four possibilities for each position, four squared): AA, BB, CC, DD, AB, BA, CA, DA, AC, BC, DC , AD, BD, CD, DC. At the end of the work, illustrations related to this combinatorial analysis are included. For example, the following illustration shows the four tilings of a single tile (which are the four possible orientations) and two tiles (horizontally), which are the ones that correspond to the previous list of sixteen.



Page from the book of the Carmelite father Dominique Doüat that contains tiling with a single tile, or with two tiles, using the Truchet tessera

The next analysis that is carried out is that of the possible tilings with three tiles (horizontally), which is equal to $4^3 = 64$ (four possibilities –A, B, C, D- for each position), as shown in Fig. following table.



UNE INFINITE' DE DESSEINS. 9

TROISIE'ME TABLE,
Contenant 64. Permutations.

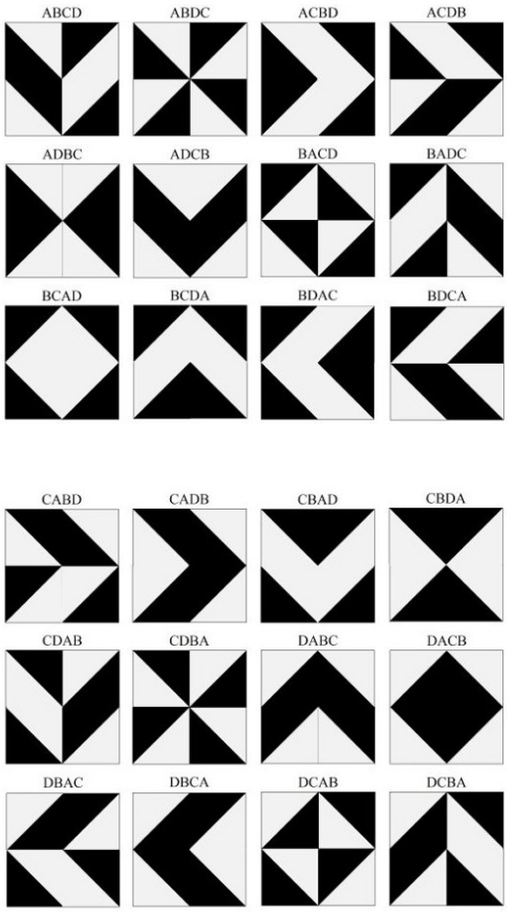
1 AAA	2 BBB	33 CCC	34 DDD
3 AAB	4 BBA	35 CCA	36 DDA
5 AAC	6 BBC	37 CCB	38 DDB
7 AAD	8 BBD	39 CCD	40 DDC
9 ABA	10 BAB	41 CAC	42 DAD
11 ACA	12 BCB	43 CBC	44 DBD
13 ADA	14 BDB	45 CDC	46 DCD
15 ABB	16 BAA	47 CAA	48 DAA
17 ACC	18 BCC	49 CBB	50 DBB
19 ADD	20 BDD	51 CDD	52 DCC
21 ABC	22 BAC	53 CAB	54 DAB
23 ABD	24 BAD	55 CAD	56 DAC
25 ACB	26 BCA	57 CBA	58 DBA
27 ACD	28 BCD	59 CBD	60 DBC
29 ADB	30 BDA	61 CDA	62 DCA
31 ADC	32 BDC	63 CDB	64 DCB

B

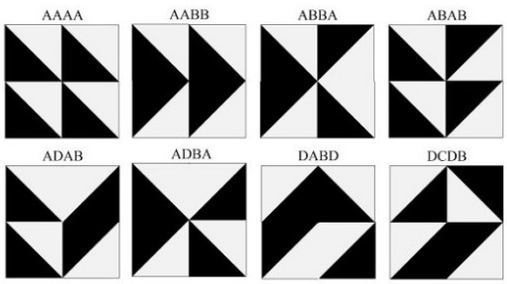
Page from Dominique Doüat's book containing tiling with three tiles using the bicolor square tessera

While the possible tiling patterns with four tiles (horizontally) are $4^4 = 256$. Let us take advantage of this analysis to comment on the possible tiling patterns of a 2 x 2 square pavement using four bicolor square tiles. If we think about it, we can see that we have four possible positions for our tiles, top left, top right, bottom left and bottom right, then it comes down to analyzing the possible ways to place Truchet tiles, with its four orientations (A, B, C and D) in those four positions. **The solution is exactly the Doüat analysis, so** the possible tessellation patterns of the 2 x 2 square pavement would also be 256.

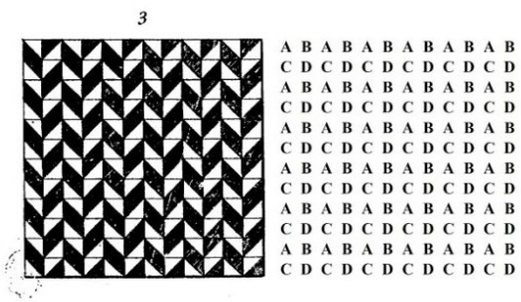
Let's look at some of these patterns. For example, if we consider tiles of the 2 x 2 square pavement with the four different tiles, that is, the four tiles appear oriented A, B, C and D, 24 different tiling patterns are obtained, the permutations of (A, B, C, D): ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB and DCBA. And the illustrations (concrete tessellations) that correspond to these letter codes are as follows.



These are the 24 tiling patterns with the four different tiles, but **if we can repeat tiles, we get to the 256 mentioned** . A few more are shown in the following image.



The larger our pavement, the more complex the analysis will be. Here are some illustrations of 12 x 12 tile tiling from the book *Method for Making Countless Different Designs Using Two-Color Squares Separated by a Diagonal Line* , along with their expression using the letters A, B, C, and D.



B B C B D B C A C B C C
B D A D B A D C A D A C
A C D B D B C A C A B D
B D B D B A D C A C A C
D B D B D C B A C A C A
B C B C A D A D B C B C
A D A D B C B C A D A D
C A C A C D A B D B D B
A C A C A B C D B D B D
B D C A C A D B D B A C
A C B C A B C D B C B D
A A D A C A D B D A D D

Regarding the general classification of tiling patterns, clearly, **the number of different patterns for a pavement with n tiles is 4^n**, which we can describe with the corresponding strings of letters.

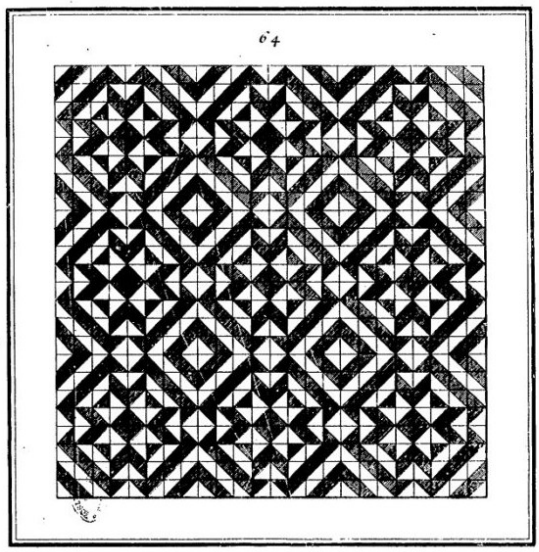


Illustration of a tiling pattern, with a Truchet tile, from the book *Method for making an infinity of different designs with squares of two colors separated by a diagonal line* (1722), by Dominique Doüat

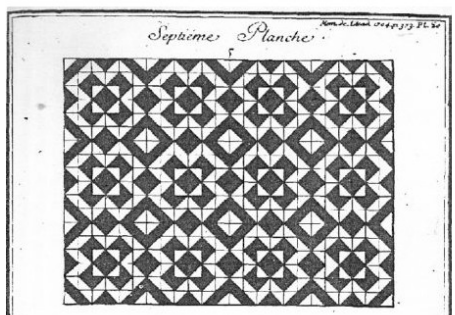
The tiled floors of Truchet

Although research work on tiling patterns with the simple square tessera of two colors divided by the diagonal was started by the French Dominican priest Sebastien Truchet (1657-1729) in his publication *Memory on Combinations* , published in *Histoire de l'Académie Royale des Sciences de Paris* , in 1704.

Sebastien Truchet himself explains at the beginning of his work how it occurred to him to start investigating the combinatorics of these tiling patterns:

> *During the last trip I made to the Orleans canal by order of His Royal Highness, at a castle called Motte St. Lye, 4 leagues this side of Orleans, I found several ceramic tiles that were intended to tile the floor of a chapel and from several other apartments. They were square in shape, divided by a diagonal line into two colored parts. In order to form nice designs and patterns by arranging these tiles, I first examined the number of ways these tiles could be joined in pairs, always in a checkerboard arrangement [...].*

From the historical point of view we must bear in mind that, although some concepts and techniques of combinatorics have been studied since ancient times, the origin of modern combinatorics can be established in the 17th and 18th centuries, thanks to the work of mathematicians such as Frenchman Blaise Pascal (1623-1662), German Gottfried Wilhelm Leibniz (1646-1716), who published the text *De Arte Combinatoria*(1666), the British Isaac Newton (1643-1727), the Swiss Jacob Bernoulli (1655-1705) or the Swiss Leonhard Euler (1707-1783), who initiated or developed the study of many areas of combinatorics (the theory of graphs, Greco-Latin squares, partitions, the problem of the horse's path, etc), among others. Therefore, the works of Sebastien Truchet and Dominique Doüat appeared in full development of the theory of combinatorics.



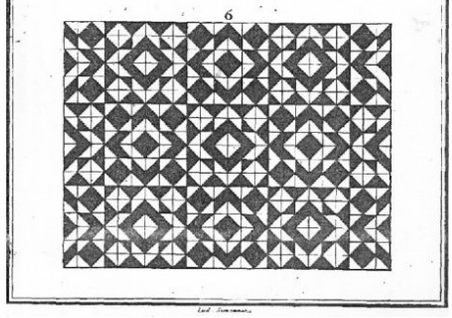
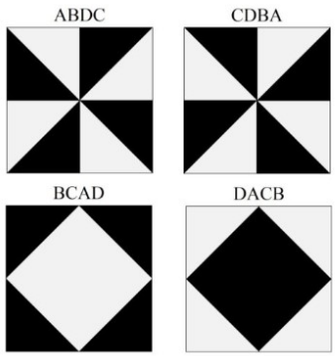


Illustration of a tiling pattern, with a Truchet tile, from the publication *Memory on Combinations* (1704), by Sebastien Truchet

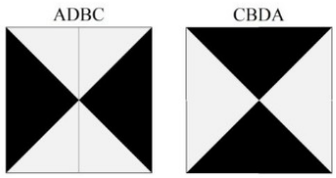
Clearly, the best way to achieve beautiful patterns -as we can see, for example, in many texts on quilt designs- is to work directly with the two-tone square tiles in all four orientations (A, B, C and D), but **from the From a mathematical point of view, working with the letters, as Doüat did, allows a better analysis of the combinatorial possibilities and helps to distinguish one from the other** , since the list of letters becomes the code that allows us to identify each tiling.

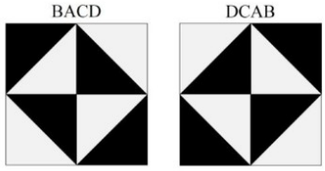
Quilt made with "Truchet squares" -or triangles of half squares, as they are called in quilt manuals-. Carried Away Quilting Page Image

Another question that we can take into account when studying tessellation patterns is **what type of symmetry they have** . For example, if we continue with the 2 x 2 square pavements, although we only pay attention to the 24 tiling patterns (of the 256 in total) with the four different tiles that we have described above, we can observe that there are different types of symmetries . For example, the ABCD, CDBA, BCAD and DACB patterns remain invariant, that is, they do not change, even if we rotate them by 90, 180 or 270 degrees. They have a rotational symmetry of 90 degrees.

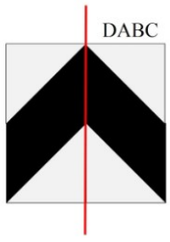


The ADBC, CBDA, BACD and DCAB patterns are kept invariant by 180 degree **rotations** . They have a rotational symmetry of 180 degrees. Although we note that if we make a 90-degree turn, we do not get the same pattern, but a similar pattern with a change in colors (turning ADBC 90 degrees gives CBDA and vice versa, and turning BACD 90 degrees gives DCAB and vice versa, which are patterns with the colors changed).

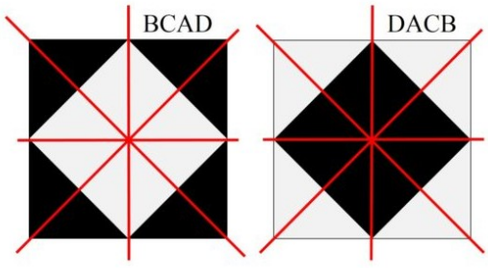




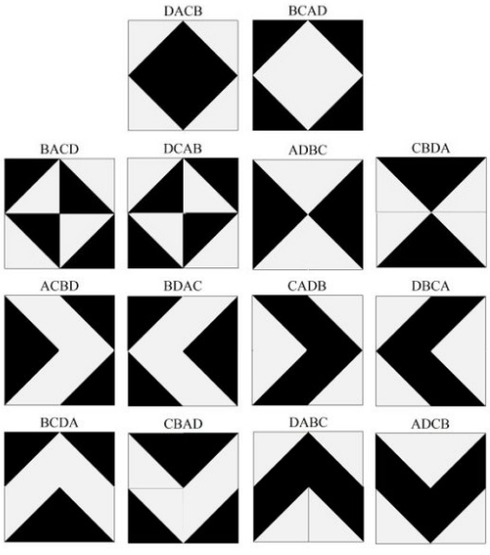
Another type of symmetries that we can consider are the **symmetries (specular) with respect to a line** , that is, what is on one side and the other with respect to the line is the same (its mirror image, as if the line were the mirror). For example, the DABC pattern has symmetry about the vertical line through the center, as shown in the image.



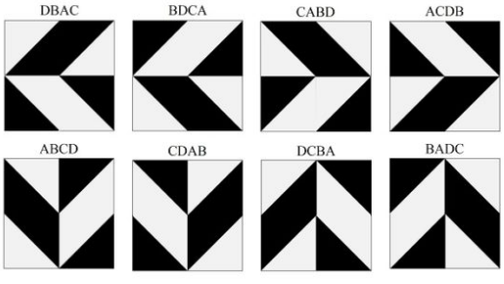
In this sense, we have that the DACB and BCAD patterns, which we know to have a rotational symmetry of 90 degrees from what we have seen above, have mirror symmetry with respect to four lines that pass through the center of the square, the two horizontal and vertical lines, thus as the two diagonals of the square.



The BACD and DCAB patterns have mirror symmetry with respect to the two diagonals, the ADBC and CBDA patterns with respect to the vertical and horizontal lines that pass through the center of the square, the ACBD, BDAC, CABD and DBCA patterns with respect to the horizontal line that passes through the center and BCDA, CBAD, DABC and ADCB with respect to the vertical passing through the center. In total, fourteen patterns are mirror symmetry with respect to lines, which appear in the following image.



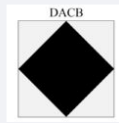
In short, of the 24 tiling patterns with the four different tiles, we find that 16 have some rotational or mirror symmetry, which are the "normal symmetries", while there are 8 patterns (the ones we see in the following image) that do not have any rotational or mirror symmetry. none of these symmetries.



Seen together these 8 patterns, we can perceive that they have a certain type of symmetry. Specifically, they have **symmetry with color change** . The first four patterns have a mirror symmetry with respect to the horizontal line that passes through the center, but with a change in color, that is, the areas that are white on one side, black on the opposite side, and vice versa. While the last four patterns have a mirror symmetry with respect to the vertical line that passes through the center.

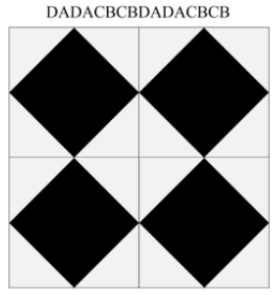
In conclusion, all tessellation patterns for a square pavement of size 2 x 2 have normal symmetry or symmetry with color change. This is precisely the simple version of what is known as the diamond theorem, which we describe below.

diamond theorem

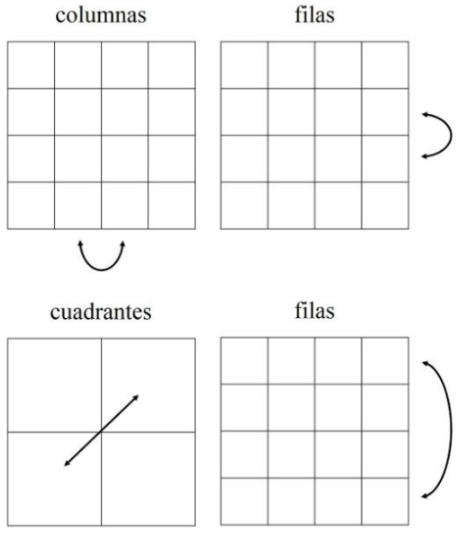
> ***Diamond theorem (2 x 2 version)*** *: Let D be the figure formed by the black diamond, constructed with the four different orientations of the Truchet bicolor tile, shown in the following image:*
>
> 
>
>
> *If G is the group of 24 permutations of the four squares (tiles) of D, then the image of D by any of the elements of the group G gives rise to a figure that has normal symmetry or symmetry with color change.*

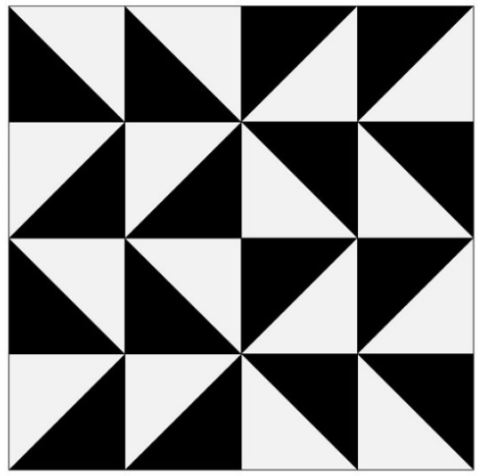
But **this is only a simple version, which will allow us to better understand the normal version** of the true diamond theorem, which we explain below.

Let D now be the figure of four black diamonds on a 4 x 4 square that we show in the following image and whose letter code is DADACBCBDADACBCB,



and let G be the group of 322,560 transformations generated by the permutations of any two rows of the 4 x 4 square, the permutations of any two columns, or the permutations of any two 2 x 2 quadrants. For example, if we apply the permutation of the middle columns to figure D, then the permutation of the middle rows, then the permutation of the lower left and upper right quadrants, and finally the permutation of the two outer rows, the result is the following figure.



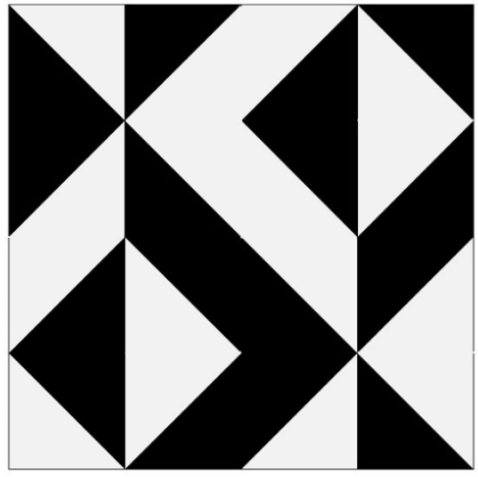


The diamond theorem, by the American mathematician Steven H. Cullinane (1942), says the following.

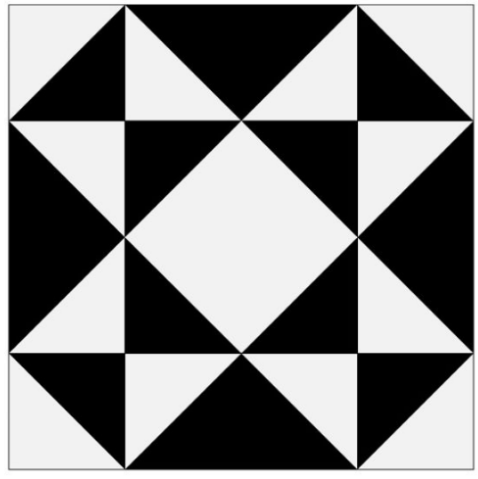
> ***Diamond theorem*** *: All the images obtained by any of the 322,560 transformations of the group G of figure D have normal symmetry or symmetry with color change.*

For example, the previous image has mirror symmetry with color change with respect to the horizontal and vertical lines that pass through the center.

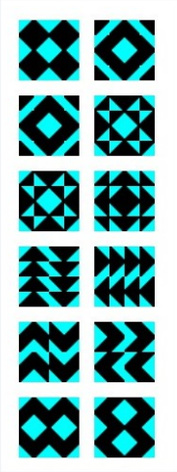
Or the following figure obtained by permuting the first two rows, then the two columns on the right, then the upper left quadrant with the lower right, and finally the two central columns, has a rotational symmetry of 180 degrees with color change.



Or, for example, the figure below has both rotational and mirror symmetries.



But we have not said how to obtain the previous figure from the original figure of the four black diamonds. This is a problem, or a diversion, that Steven H. Cullinane himself poses as a puzzle and that he baptizes with the name "diamond 16 puzzle".

> *Diamond puzzle 16 : First build 16 two-color Truchet tiles on cardboard, obtain one of the figures in the left column, in the following image, and try to transform it into the corresponding figure in the right column, using the transformations of the theorem of the diamond, that is, permutations of any two rows, any two columns, or any two 2 x 2 quadrants.*
>
> 

If you don't want to build the Truchet tiles and play the 16 diamond puzzle with them, you can do it in the **online version** that is included in the bibliography.

We end with the image of a quilt made using a tessellation design with the bicolor square structure, with the two colors separated by the diagonal, the Truchet tile.

Quilt titled "Blue tango" from Louisa Enright 's blog

Bibliography:

1.- Cyril Stanley Smith (with translation of Truchet's text by Pauline Boucher), *The Tiling Patterns of Sebastian Truchet and the Topology of Structural Hierarchy* , Leonardo, vol. 20, no. 4, p. 373-385, 1987.

2.- Dominique Doüat, *Méthode pour faire une infinité desseins différents avec des carreaux mi-partis de deux couleurs par une ligne diagonale: ou observations du Père Dominique Doüat, Religieux Carme de la Province de Toulouse, sur un mémoire inséré dans l' Histoire de l'Académie Royale des Sciences de Paris l'année 1704, presented by the Révérend Père Sébastien Truchet religieux du même ordre, Académicien honoraire [Method for making an infinity of different designs with squares of two colors separated by a diagonal line*], Paris, 1922.

Facsimile (excerpts) and introduction by Jacques André

Complete work in Gallica – Bibliothèque nationale de France

3.- Sebastien Truchet, *Memoir sur les Combinaisons* , *Histoire de l'Académie Royale des Sciences de Paris* , 363-372 (1704).

Complete work in Gallica – Bibliothèque nationale de France

4.- Steven H. Cullinane, *The Diamond Theorem* , eprint arXiv:1308.1075 , 2013.

5.- Steven H. Cullinane, The Diamond Theorem

6.- 5.- Steven H. Cullinane, The Diamond 16 puzzle

About the author: Raúl Ibáñez is a professor in the Department of Mathematics at the UPV/EHU and collaborator with the Chair of Scientific Culture

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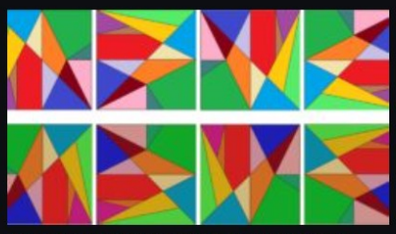
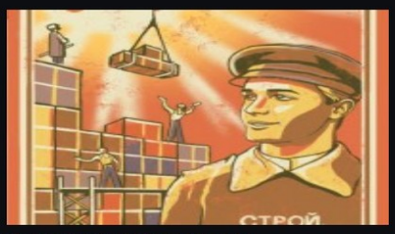


The Stomachion puzzle and the Archimedean palimpsest (2)

In the first entry of this mini-series of the Mathemotion section of the Scientific Culture Notebook, The Stomachion puzzle and [...]

The Stomachion puzzle and the Archimedean palimpsest (1)

In this mini-series of two entries of the Scientific Culture Notebook I would like to talk about the Tangram-type geometric puzzle [...]



Tetris, tiling and demonstrations

In one of the recent entries in the Mathemotion section of the Scientific Culture Notebook, "Tiling with L-triominos (An example [...]

1 COMMENT

The art of the simple Truchet tile — Notebook of Scientific Culture
[...] the previous entry of the Notebook of Scientific Culture Truchet tiling and the diamond puzzle we were talking about the design of tiling patterns made with a simple tile, the [...]

LEAVE A REPLY

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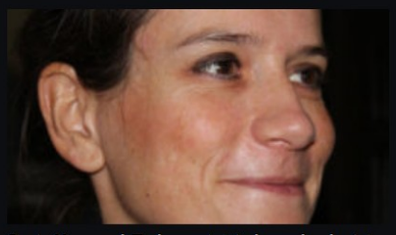
Other projects

ZIENTZIA KAIERA



Sonia Harmand, Turkana aintzirako archeologist

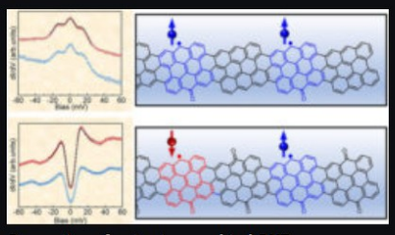
Carolina Martínez Sonia Harmand zientzialari frantsesa ospe handiko aditua da Harri Aro goiztiarreko arkeologiaren esparruan. Harmand, 1974an jaioa, ikerketa bateko [...]

WOMEN WITH SCIENCE

Women in science, technology, engineering and mathematics

The Women in STEM initiative (Canada's Ingenium Program) aims to "engage, [...]

MAPPING IGNORANCE



Generation of net spins on chiral GNRs

Graphene nanoribbons (GNRs), are strips of graphene with ultra-thin width (1 GNRs are very interesting structures, partly due to their [...]

Information

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